Class Note 25: Second-Order Circuits

A. Preface

1. A second-order circuit is a circuit environment where an inductor and a capacitor are present simultaneously.
2. The second-order circuit analysis is, in this class, is limited to one loop (series RLC) or one non-reference node (parallel RLC) case.
3. PSPICE analysis practice is encouraged.

B. Solution of a Second-Order differential Equation (part 1: solution for forced function)

1. Let's change equations (1) and (2) to a more general second-order differential equation form:

$$\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_2x(t)=K \quad \text{--------------------------------(3)}$$

2. As we did in the first-order analysis, the solution of the equation (3) is:

$x(t)=x_p(t)+x_c(t)$

where, $x(t)=x_p(t)$ is a solution to $\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_2x(t)=K$ -------------(3a)

and $x(t)=x_c(t)$ is a solution to $\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_2x(t)=0$ ----------------------(4)

3. Let's observe equation (3a) for a while. Since the right hand side is a constant K, therefore $x_p(t)$ must be a constant (at left hand side). Let say $x_p(t)=C$ (C is a constant), then the left hand side is: a_2C. Therefore, $x_p(t)=C=\frac{K}{a_2}$.
4. The, the complete solution of equation (3) is of the form:

$$x(t)=x_p(t)+x_c(t)=\frac{K}{a_2}+x_c(t)$$

5. The solution of the homogeneous equation for $x_c(t)$ [eq. (4)] starts in the next section.

C. Solution of a Second-Order differential Equation (part 2: solution for homogeneous eq.)

1. For simplicity (you will see why soon), let's rewrite the equation (4), by simple substitutions for $a_1=2\alpha$ and $a_2=w_o{}^2$, in the form of:

$$\frac{d^2x(t)}{dt^2}+2\alpha\frac{dx(t)}{dt}+w_o{}^2x(t)=0 \quad \text{---------------------------------------(5)}$$

*Note: For this revised equation form, $x_p(t)=\frac{K}{w_o^2}$ since $a_2=w_o{}^2$.

2. We assume a solution that: $x_c(t)=Ae^{st}$
3. The substitution of the assumed solution into equation (5) yields

$$s^2Ae^{st}+2\alpha sAe^{st}+w_o^2Ae^{st}=0$$

4. Simplification of the above equation yields to: $(s^2+2\alpha s+w_o{}^2)Ae^{st}=0$

Since $x_c(t)=Ae^{st}$ cannot be zero, $s^2+2\alpha s+w_o{}^2=0$ ------------------------(6)

5. The equation (6) is called the <u>characteristic equation</u>, where

α is referred to *Neper Frequency*
w_o is referred to *Undamped Natural Frequency* (or *Resonant Frequency*)
and $\left(\frac{\alpha}{w_o}\right)$ is referred to *Exponential Damping Ratio*.

6. If the characteristic equation is satisfied, then, the assumed solution $x(t) = Ae^{st}$ is correct.
7. Employing the quadratic formula, the roots for the characteristic equation are:

$$s = \frac{-2\alpha \pm 2\sqrt{\alpha^2 - w_o^{\;2}}}{2} = -\alpha \pm \sqrt{\alpha^2 - w_o^{\;2}}$$

8. Therefore, the two roots are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_o^{\;2}} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - w_o^{\;2}}$$

9. This means that we have two solutions of the homogeneous equation:

$$x_1(t) = A_1 e^{s_1 t} \text{ and } x_2(t) = A_2 e^{s_2 t}$$

10. Note that the sum of two solutions is also a solution. Therefore, in general the solution of the homogeneous equation is of the form:

$$x_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \text{ ---(7)}$$

11. Finally, the solution for the original second-order equation of

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + w_o^{\;2}x(t) = K$$

is:

$$\underline{x(t) = \frac{K}{w_o^{\;2}} + x_c(t) = \frac{K}{w_o^{\;2}} + A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

with, $s_1 = -\alpha + \sqrt{\alpha^2 - w_o^{\;2}}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - w_o^{\;2}}$.

12. From the solution, we can easily see the final value: $x(\infty) = \frac{K}{w_o^{\;2}}$.

D. Examination of the solution of the homogeneous equation: Natural Frequency Analysis

1. Let's have a closer examination of the roots of the characteristics roots:

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_o^{\;2}} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - w_o^{\;2}}$$

2. The roots s1 and s2 are called the natural frequencies because they determine the natural (unforced) response of the network.
3. We see that the roots are dependent upon the value of $(\alpha^2 - w_o^{\;2})$.
4. If $\alpha^2 = w_o^{\;2}$: the roots are ***real and equal*** --> "Critically Damped"
   If $\alpha^2 > w_o^{\;2}$: the roots are ***real and unequal*** --> "Overdamped"
   If $\alpha^2 < w_o^{\;2}$: the roots are ***complex numbers*** --> "Underdamped"

5. "Critically Damped" case: (real and equal ***s***)

(a) Condition: $\alpha^2 = w_o^{\ 2} \Rightarrow \underline{s_1 = s_2 = -\alpha}$

(b) Solution Form: $x(t) = x(\infty) + D_1 te^{-\alpha t} + D_2 e^{-\alpha t}$

[Note: $x(t) = x(\infty) + A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = (A_1 + A_2)e^{-\alpha t} = A_3 e^{-\alpha t}$. This simple form, however, in general does not satisfy the two initial conditions, i.e., $x(0)$ and $\left.\frac{dx(t)}{dt}\right|_{t=0}$ with the single constant A_3. After applying an approach for repeated roots, the solution for critically damped case is of the form: $x(t) = x(\infty) + A_3 e^{-\alpha t}(A_1 + A_2 t)$]

(c) Constraints (equations to find the two coefficients, D_1, and D_2):

i) $x(0) = x(\infty) + D_2$

ii) $\left.\frac{dx(t)}{dt}\right|_{t=0} = D_1 - \alpha D_2$

6. "Overdamped" case: (real and unequal ***s*****)**

(a) Condition: $\alpha^2 > w_o^{\ 2}$

(b) Solution: $x(t) = x(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

(c) Constraints (or equations to find two coefficients A_1 and A_2)

i) $x(0) = x(\infty) + A_1 + A_2$

ii) $\left.\frac{dx(t)}{dt}\right|_{t=0} = s_1 A_1 + s_2 A_2$

7. "Underdampled" case: (complex ***s*****)**

(a) Condition: $\alpha^2 < w_o^{\ 2}$. Also define $w_d = \sqrt{w_o^{\ 2} - \alpha^2}$

The roots are rewritten as:

$$s_1 = -\alpha + j\sqrt{w_o^{\ 2} - \alpha^2} = -\alpha + jw_d \quad \text{and} \quad s_2 = -\alpha - j\sqrt{w_o^{\ 2} - \alpha^2} = -\alpha - jw_d$$

Then, the solution can be rewritten as:

$$\begin{aligned}
x(t) &= x(\infty) + A_1 e^{-(\alpha - w_d)t} + A_2 e^{-(\alpha + jw_d)t} \\
&= x(\infty) + e^{-\alpha t}[A_1 e^{jw_d t} + A_2 e^{-jw_d t}] \\
&= x(\infty) + e^{-\alpha t}[A_1\{\cos w_d t + j\sin w_d t\} + A_2\{\cos w_d t - j\sin w_d t\}] \\
&= x(\infty) + e^{-\alpha t}[(A_1 + A_2)\cos w_d t + j(A_1 - A_2)\sin w_d t] \\
&= x(\infty) + e^{-\alpha t}[B_1 \cos w_d t + B_2 \sin w_d t]
\end{aligned}$$

(b) The solution form: $x(t) = x(\infty) + e^{-\alpha t}[B_1 \cos w_d t + B_2 \sin w_d t]$

(c) Constraints (Coefficient equations)

i) $x(0) = x(\infty) + B_1$

ii) $\left.\frac{dx(t)}{dt}\right|_{t=0} = -\alpha_1 B_1 + \omega_d B_2$

Second-Order Equation Summary Table

<table>
<tr><td colspan="2">Second-Order Differential Equation</td><td colspan="3">$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0{}^2 x = K$$</td></tr>
<tr><td colspan="2">Final Value</td><td colspan="3">$$x(\infty) = \frac{K}{w_o{}^2}$$</td></tr>
<tr><td colspan="2">Characteristics Roots</td><td colspan="3">$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0{}^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0{}^2}$</td></tr>
<tr><td rowspan="4">S o l u t I o n s</td><td>Damping Types</td><td>Overdapmed case</td><td>Underdamped case</td><td>Critically Damped case</td></tr>
<tr><td>Condition</td><td>$\alpha^2 > \omega_0{}^2$</td><td>$\alpha^2 < \omega_0{}^2$</td><td>$\alpha^2 = \omega_0{}^2$</td></tr>
<tr><td>Solution Form $x(t) =$</td><td>$x(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$</td><td>$x(\infty) + B_1 e^{-\alpha t} \cos \omega_d t$
$+ B_2 e^{-\alpha t} \sin \omega_d t$</td><td>$x(\infty) + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$</td></tr>
<tr><td>Coefficient Determination Relationship</td><td>$x(0) = x(\infty) + A_1 + A_2$
$\frac{dx(t)}{dt}\Big|_{t=0} = s_1 A_1 + s_2 A_2$</td><td>$x(0) = x(\infty) + B_1$
$\frac{dx(t)}{dt}\Big|_{t=0} = -\alpha B_1 + \omega_d B_2$
$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2}$</td><td>$x(0) = x(\infty) + D_2$
$\frac{dx(t)}{dt}\Big|_{t=0} = D_1 - \alpha D_2$</td></tr>
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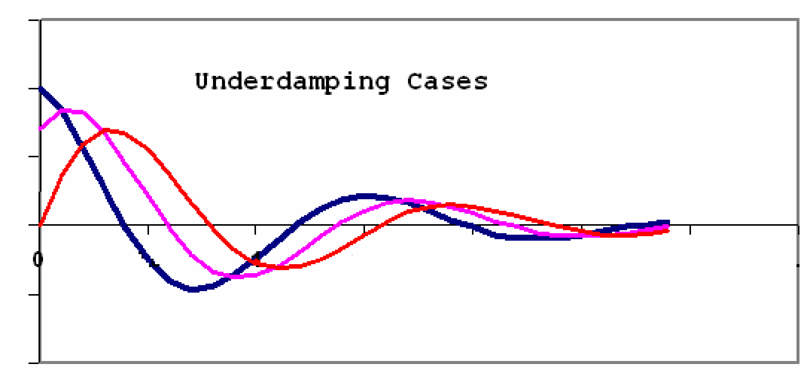
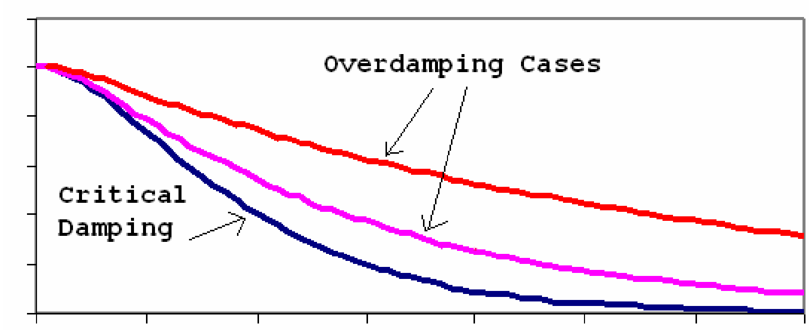
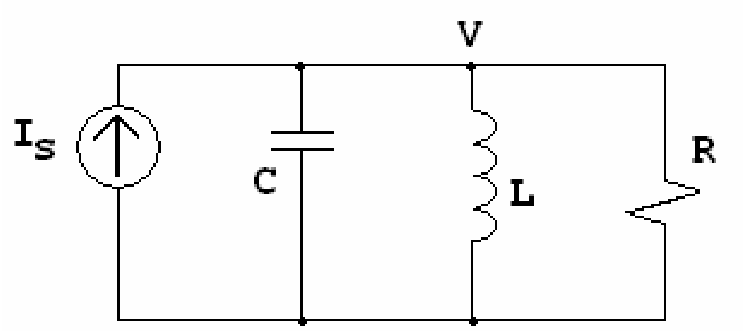




E. Basic Circuit Equation of Second-Order Circuit

1. Let's first consider a parallel RLC circuit powered by a DC current source.



2. Let's assume that there is <u>no</u> energy initially stored in the capacitor and inductor.
3. The node voltage equation is:

$$-I_s+\frac{v}{R}+\frac{1}{L}\int_{t_o}^{t}vdx+C\frac{dv}{dt}=0\rightarrow\frac{v}{R}+\frac{1}{L}\int_{t_o}^{t}vdx+C\frac{dv}{dt}=I_s$$

4. By derivation with respect to time *t*, we have:

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{v}{L}=0 \quad \text{or} \quad \frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0 \text{ --------------(1)}$$

From the general 2nd order differential equation, $\frac{d^2x(t)}{dt^2}+2\alpha\frac{dx(t)}{dt}+w_o{}^2x(t)=K$,

<u>We see that:</u>

(a) $2\alpha=\frac{1}{RC}\rightarrow\alpha=\frac{1}{2RC}$: The damping coefficient is determined by half of 1/RC (i.e., half of the 1/τ)

(b) $w_o{}^2=\frac{1}{LC}\rightarrow w_o=\frac{1}{\sqrt{LC}}$: The Resonant frequency is determined by sqrt(LC).

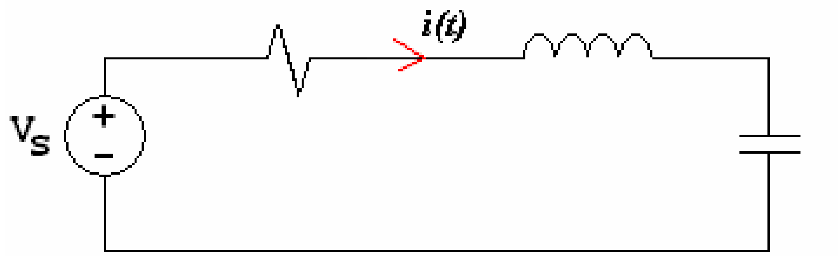
(c) Damping conditions:

i) critically damping, if $\left(\frac{1}{2RC}\right)^2=\frac{1}{LC}$

ii) over damping, if $\left(\frac{1}{2RC}\right)^2>\frac{1}{LC}$

iii) under damping, if $\left(\frac{1}{2RC}\right)^2<\frac{1}{LC}$, oscillation frequency $w_d=\sqrt{\frac{1}{LC}-\left(\frac{1}{2RC}\right)^2}$

5. Let's now consider a series RLC circuit powered by a DC voltage source.



6. Again, let's assume that there is no energy initially stored in the capacitor and inductor.
7. The loop KVL equation for the current is:

$$-V_s + Ri + \frac{1}{C}\int_{t_o}^{t} i(x)dx + L\frac{di}{dt} = 0 \rightarrow Ri + \frac{1}{C}\int_{t_o}^{t} i(x)dx + L\frac{di}{dt} = V_s$$

8. By derivation with respect to time *t*, we have:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \rightarrow \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \text{ ------(2)}$$

From the general 2nd order differential equation, $\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + w_o{}^2 x(t) = K$,

We see that:

(a) $2\alpha = \frac{R}{L} \rightarrow \alpha = \frac{R}{2L}$: The damping coefficient is determined by half of R/L (i.e., half of the $1/\tau$)

(b) $w_o{}^2 = \frac{1}{LC} \rightarrow w_o = \frac{1}{\sqrt{LC}}$: The Resonant frequency is determined by sqrt(LC).

(c) Damping conditions:

i) critically damping, if $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

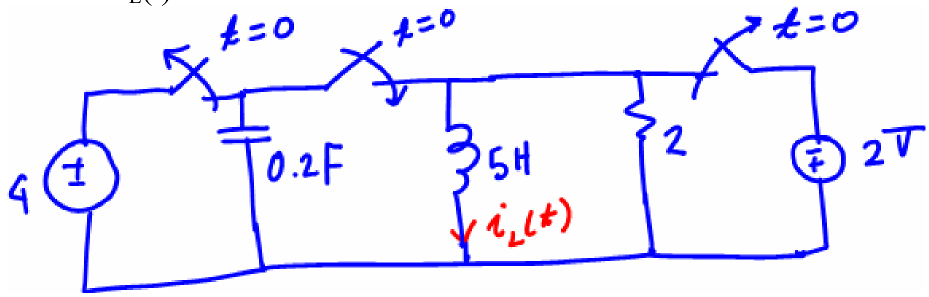
ii) over damping, if $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

iii) under damping, if $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$, oscillation frequency $w_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

9. We can see that the equation for the node voltage in the parallel RLC [equation (1) above] and the equation for the loop current in the series RLC [equation (2) above] are identical: a second-order differential equation with constant coefficients.

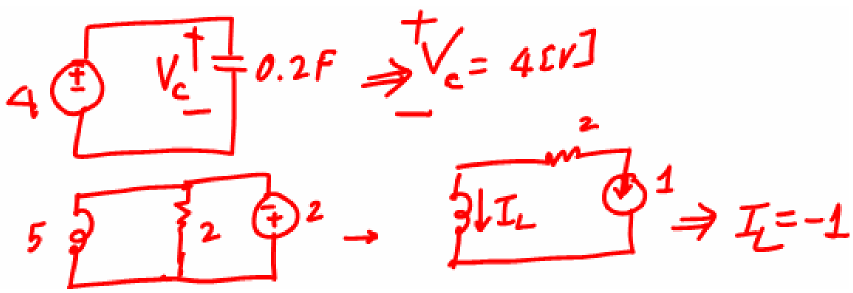
F. Parallel RLC Natural Response Example

Consider the parallel RLC circuit shown below. Find the node voltage v(t) and the current through the inductor $i_L(t)$.

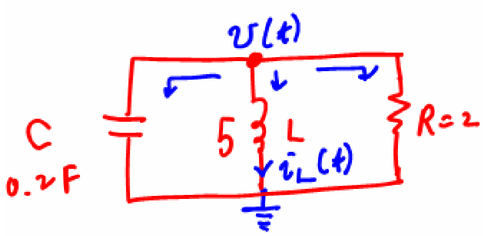


SOLUTION:

(SOL) $t<0$:



$t>0$: $V_c(0) = 4$, $i_L(0) = -1$



KCL: $\dfrac{v}{R} + C\dfrac{dv}{dt} + i_L = 0$ ①

Since $i_L = \dfrac{1}{L}\displaystyle\int_0^t v\,d\tau$

$$① \rightarrow \frac{v}{R} + C\frac{dv}{dt} + \frac{1}{L}\int_0^t v\,d\tau$$

$$\Big\downarrow \frac{d}{dt}$$

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

$$\Big\downarrow$$

$$\underline{\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0} \quad ②$$

Since $R = 2,\ C = 0.2,\ L = 5$

$$\frac{1}{RC} = \frac{1}{(2)(0.2)} = 2.5, \quad \frac{1}{LC} = \frac{1}{(0.2)(5)} = 1$$

$$② \rightarrow \frac{d^2 v}{dt^2} + 2.5\frac{dv}{dt} + 1\, v = 0$$

$$\rightarrow \quad 2\alpha = 2.5 \Rightarrow \alpha = 1.25,$$
$$\omega^2 = 1$$

Since $\alpha^2 > \omega^2$: Overdamping

Solution Form : $v(t) = \cancel{v(\infty)} + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega^2} = -1.25 + \sqrt{1.25^2 - 1^2} = -0.5$$
$$s_2 = -1.25 - \sqrt{1.25^2 - 1} = -2$$

use constraints for A_1 & A_2

(i) $v(0) = A_1 + A_2 + v(\infty)$] $\rightarrow$ $A_1 + A_2 = 4$ ①

From $v(0) = V_c(0) = 4$

(ii) $\left.\frac{dv}{dt}\right|_{t=0} = s_1 A_1 + s_2 A_2$

From ① : $\frac{v}{R} + C\frac{dv}{dt} + i_L = 0$

$$\frac{v(0)}{R} + C\left.\frac{dv}{dt}\right|_{t=0} + i_L(0) = 0 \rightarrow \left.\frac{dv}{dt}\right|_{t=0} = \frac{-\frac{v(0)}{R} - i_L(0)}{C}$$

$$= \frac{-\frac{4}{2} - (-1)}{0.2} = -5$$

$$\therefore \begin{cases} -0.5A_1 - 2A_2 = -5 & (2) \\ A_1 + A_2 = 4 & (1) \end{cases}$$

$$\left.\begin{aligned} A_1 + 4A_2 &= 10 \\ A_1 + A_2 &= 4 \end{aligned}\right\} \quad A_2 = 2, \quad A_1 = 2$$

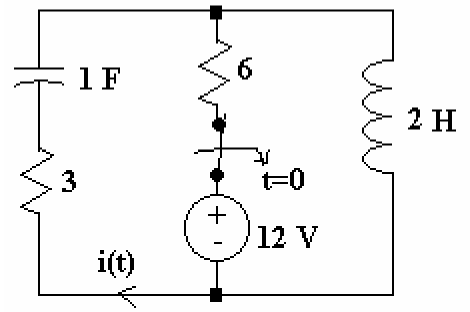
Finally

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= 2e^{-0.5t} + 2e^{-2t}$$

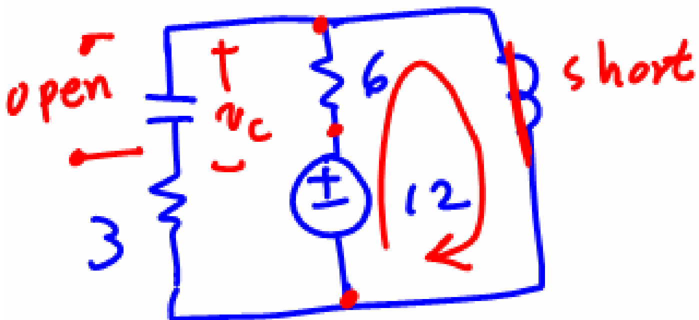
G. Series RLC Natural Response Example

The switch in the circuit has been closed for a long time. At t=0, the switch opens. Find *i*(t).



SOLUTION:

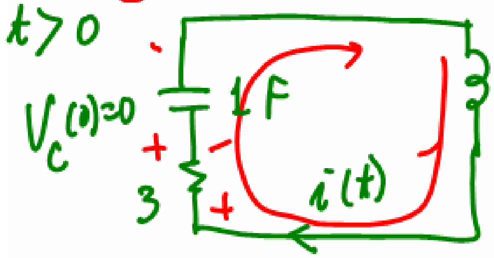
$t < 0$:



$I_L = 2[A]$

$V_c = 0 \quad (12-12=0)$

$t>0$:



KVL:

$$2\frac{di}{dt} + 3i + V_c = 0 \quad (1)$$

Since $V_c = \frac{1}{C}\int i\,d\tau$

(1) $$2\frac{di}{dt} + 3i + \frac{1}{C}\int_0^t i\,d\tau = 0$$

$$\Big\downarrow \frac{d}{dt}$$

$$2\frac{d^2 i}{dt^2} + 3\frac{di}{dt} + \frac{1}{C} i = 0$$

$$\longrightarrow \frac{d^2 i}{dt^2} + \frac{3}{2}\frac{di}{dt} + \frac{1}{2} i = 0$$

$2\alpha = \frac{3}{2} \rightarrow \alpha = 0.75$

$\omega^2 = \frac{1}{2}$

$\rightarrow \alpha^2 = \frac{9}{16} > \omega^2$

overdamping.

$$i(\infty) = \frac{K}{\omega^2} = 0$$

$$\therefore \; i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega^2} = -0.75 + \sqrt{(0.75)^2 - \tfrac{1}{2}} = -0.5$$

$$s_2 = -\alpha + \sqrt{\alpha^2 - \omega^2} = -1$$

$$= \underbrace{A_1 e^{-0.5t} + A_2 e^{-t}}$$

use constraints to find A_1 & A_2

(i) $i(0) = \underline{A_1 + A_2 = 2}$ ① ← (from $t<0$ analysis)

(ii) $\left.\frac{di}{dt}\right|_{t=0} = s_1 A_1 + s_2 A_2 = -0.5A_1 - A_2 = -3$ ②

From

$$2\frac{di}{dt} + 3i + v_c = 0 \quad ①$$

$$\left.\frac{di}{dt}\right|_{t=0} = \frac{-3i(0) - v_c(0)}{2} = \frac{-3\cdot 2 - 0}{2} = -3$$

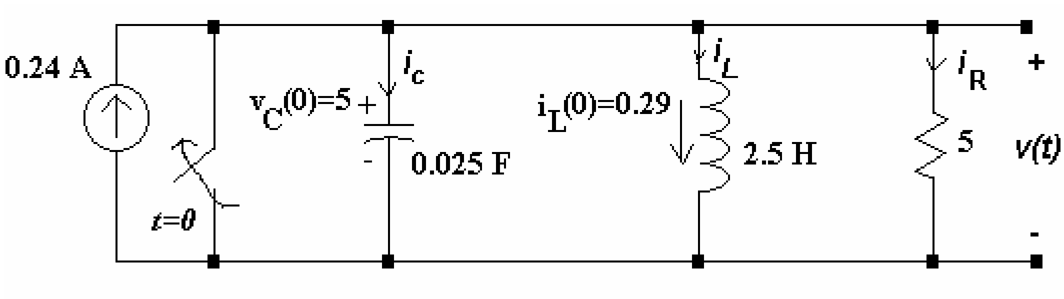
From ① & ②

$$\left.\begin{aligned} A_1 + A_2 &= 2 \\ -0.5A_1 - A_2 &= -3 \end{aligned}\right\} \quad A_1 = -2,\ A_2 = 4$$

Finally, $i(t) = -2e^{-0.5t} + 4e^{-t}$

H. Step Response of Parallel RLC Example

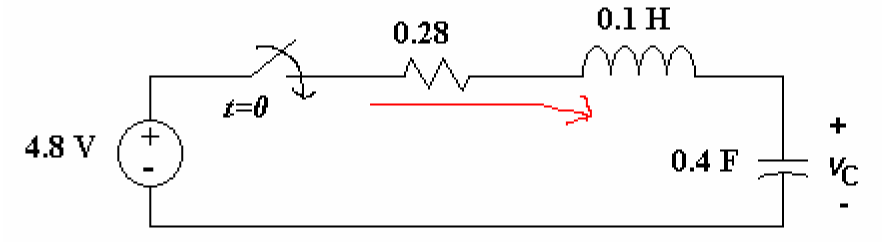
Energy is stored in the circuit before the DC current source is applied, with $i_L(0) = 0.29$ [A] and $v_C(0) = 5$ [V]. Find $i_L(t)$.



SOLUTION

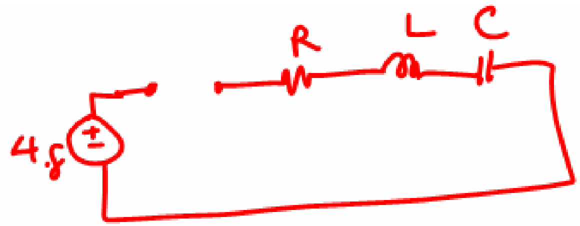
I. Step Response of Series RLC Example

Find $v_C(t)$.



SOLUTION:

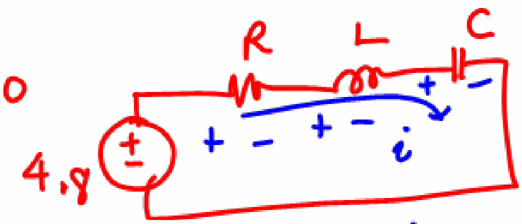
$t<0$:



No charge.

$I_L(0)=0, \quad V_c(0)=0$

$t>0$



KVL

$$-4.8 + Ri + L\frac{di}{dt} + v_c = 0 \quad (1)$$

Since $v_C = \frac{1}{C}\int_0^t i\,d\tau$

$$(1) \rightarrow \quad Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau = 4.8$$

$$\Downarrow \frac{d}{dt}$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}\left[i(t) - \cancelto{0}{i(0)}\right] = 0$$

$$\rightarrow \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0 \Rightarrow \frac{d^2 i}{dt^2} + \frac{0.28}{0.1}\frac{di}{dt} + \frac{1}{(0.1)(0.4)} i = 0$$

$$\frac{d^2 i}{dt^2} + 2.8\frac{di}{dt} + 25 i = 0$$

$$2\alpha = 2.8 \longrightarrow \alpha = 1.4$$

$$\omega^2 = 25$$

α^2 vs ω^2 $\longrightarrow$ $\alpha^2 < \omega^2$ Underdamping

$$i(t) = i(\infty) + e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

where $\omega_d = \sqrt{\omega^2 - \alpha^2} = \sqrt{25 - 1.4^2} = 4.8$

$$i(\infty) = \frac{K}{\omega^2} = 0$$

$$= e^{-1.4t}(B_1 \cos 4.8t + B_2 \sin 4.8t)$$

Use constraints to find B_1 & B_2:

(i) $i(0) = i(\infty) + B_1 = 0$ $\therefore B_1 = 0$

(ii) $\left.\frac{di}{dt}\right|_{t=0} = -\alpha B_1 + \omega_d B_2$ what is $\left.\frac{di}{dt}\right|_{t=0} = ?$

From

① $-4.8 + Ri + L\frac{di}{dt} + v_c = 0$

$$\left.\frac{di}{dt}\right|_{t=0} = \frac{4.8 - Ri(0) - v_C(0)}{L} = \frac{4.8}{L} = \frac{4.8}{.1} = 48$$

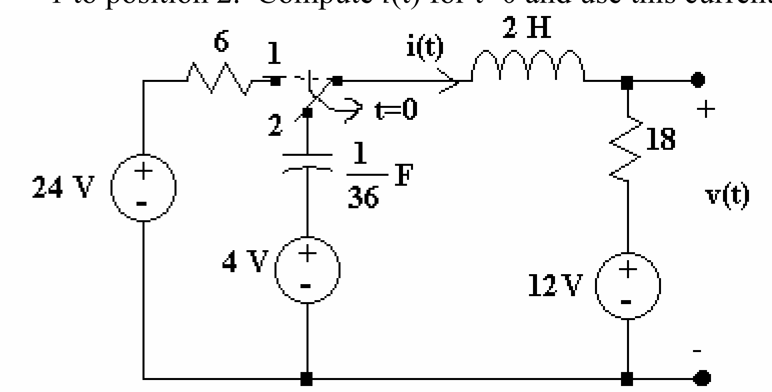
$\therefore \quad -1.4 \cancel{B_1} + 4.8 B_2 = 48 \rightarrow \underline{B_2 = 10}, \underline{B_1 = 0}$

$\underline{\text{Finally}}$.

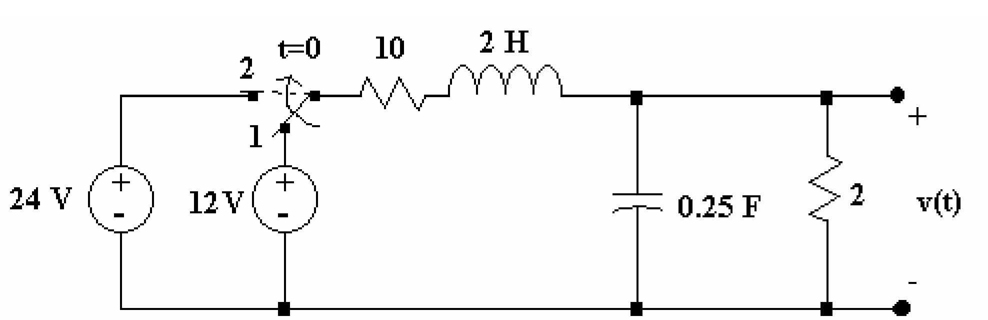
$$i(t) = 10 \cdot e^{-1.4t} \cdot \sin 4.8t$$

J. RLC Response Extra Problems

1. The switch in the circuit has been in position 1 for a long time. At t=0, it moves from position 1 to position 2. Compute *i*(t) for t>0 and use this current to determine the voltage *v*(t).



2. The switch in the circuit has been in position 1 for a long time. At t=0, it moves from position 1 to position 2. Compute *v*(t) for t>0.



3. Find v(t) and i(t)